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ORIGINAL PAPER

A fluctuation removal based univariate integration over prescribed nodes

One node fluctuation free integration under higher order set of conditions with an extension to multinode case

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Abstract This work aims at the evaluation of a univariate integral based on one node fluctuation free integration and by using exponential weight function. This is the companion of a paper presenting fundamental aspects of one node integration. In contrast to companion paper, this paper deals with more than one conditions. We construct weight function generating subspace which has three parts: (1) Eigenpolynomials, (2) Principle polynomials, (3) Unaffecting polynomials. The exponential weight function is used as the most important agent to this end. An extension to multinode case is also given.

Keywords Univariate integration · Quadrature · Fluctuation free representation · Weight functions

1 Introduction

In the companion of this paper [10], we gave certain important aspects of one node fluctuation free integration. There, we cited various previous works [1–9], those we find important to build a powerful background for our new approach to get numerical

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value of a given univariate integral [10]. We also give sufficient information about fluctuation free integration there.

Our basic idea has been based on the construction of a weight function generating subspace in the Hilbert space where the function to be integrated lies. Our investigations showed that polynomial basis function construction for this subspace is not helpful since it brings restrictions on the location of the single node. Hence we sought nonpolynomial structures and found that an exponential weight function gives a restriction free structuring. We also sought the chance of using some other weight functions instead of unit constant function and found that the allowed nodal zones can be shifted to one of the endpoints of the integration interval. At the final point of our investigations, we became oriented to use exponential weight function.

The reason why we intend to use one node integration is itis simplicity, and as we noted later, the possibility of removing more powers of deviations from a mean value by using weight function generating subspaces. This work is devoted to this goal.

On the other hand, a multinode integration can be partitioned to multi one-node-inegration because of the fundamental features of integration. Hence, one can formulate a one-node-integration based scheme for a multinode integration.

The paper is organised as follows. The second section gives certain spectral properties of the exponential weight. The third section is devoted to the construction of exponential weight function generators for one node case under more than one condition. The fourth section contains some numerical examples. In the fifth section characteristic subspace spanned by eigenpolynomials are given. The sixth section mentions about the multinode fluctuation free integration, while the seventh section finalizes the paper with the concluding remarks. Certain illustrative implementations are distributed throughout the sections.

2 Certain spectral properties of the exponential weight

In the previous paper [10], we have observed that the exponential weight function $\exp(\alpha x)/\ (\exp(\alpha) - 1)$ has some powerful properties for our purposes. First of all, it can produce any node on the interval [0, 1] including its endpoints by appropriately choosing its exponential parameter α. The α value corresponding to a specific node is unique because of the nature of the exponential weight function. We will use these nice properties in our one node and then multinode fluctuation free integrations.

We are going to investigate the spectral behaviors of the universal matrix $\mathbf{X}^{(n)}(\alpha)$ here in a little bit details to clarify what we are going to do in the construction of the weight function generating space. We are giving three figures for the eigenvalues of this universal matrix. Each eigenvalue is depicted with respect to increasing α values and all eigenvalues for a specific subspace dimension n are given in the same figure. Figures 1, 2, and 3 are corresponding to the $n = 2, 3, 4$ values, respectively. We used computer algebraic system MuPAD's plotting facilities. Although we have not given the explicit proof, the eigenvalues of the universal matrix, that is, the matrix representation of the algebraic independent variable operator which multiplies its argument by the independent variable over n dimensional subspace of the Hilbert space spanned by the orthonormal polynomials on the interval [0, 1] and under the

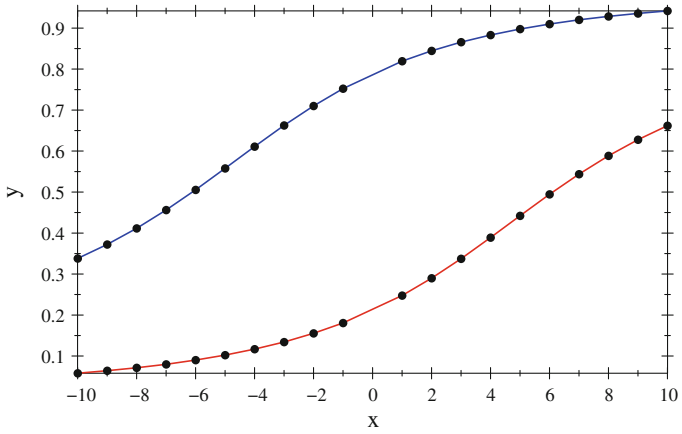

Fig. 1 Eigenvalues of **X** for $n = 2$

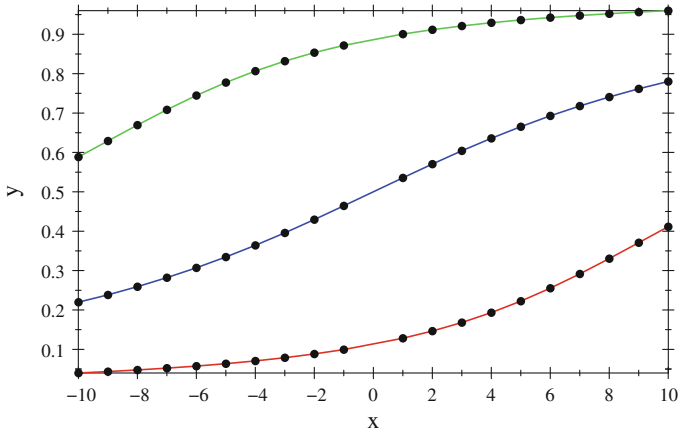

Fig. 2 Eigenvalues of **X** for $n = 3$

exponential weight function mentioned above, accumulates at 1 when α goes to infinity. Similarly, unboundedly diminishing α values cause the accumulation of the eigenvalues at 0 when α goes to minus infinity. In figures, exponential parameter domain is chosen between -10 and 10 inclusive. The universal matrix $\mathbf{X}^{(n)}(\alpha)$ is the matrix representation of the independent variable multiplication operator $\widehat{x}$'s restriction on the subspace spanned by the basis set $\mathcal{U}_n$ whose elements $u_1(x, \alpha)$, …, $u_n(x, \alpha)$ are obtained from the orthonormalization of the power set x^0, …, x^{n-1} over the unit interval [0, 1] and under the exponential weight function $\exp(\alpha x)/(\exp(\alpha) - 1)$. Hence its eigenvectors, $\mathbf{x}_i(\alpha)$s, are the linear combination coefficients of the operator $\widehat{x}_{res}^{(n)}$ (which can also be expressed as $\widehat{P}^{(n)}\widehat{x}\widehat{P}^{(n)}$ where $\widehat{P}^{(n)}$ projects from the entire Hilbert space to its subspace spanned by $\mathcal{U}_n$) in terms of $u_i(\alpha)$s. If we denote the eigenpolynomials of $\widehat{x}_{res}^{(n)}$ by $p^{(eig)}(x, \alpha)$ then we can write

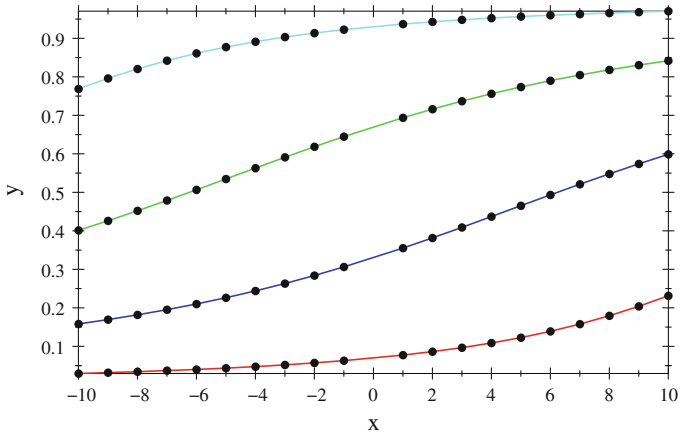

Fig. 3 Eigenvalues of $\mathbf{X}$ for $n = 4$

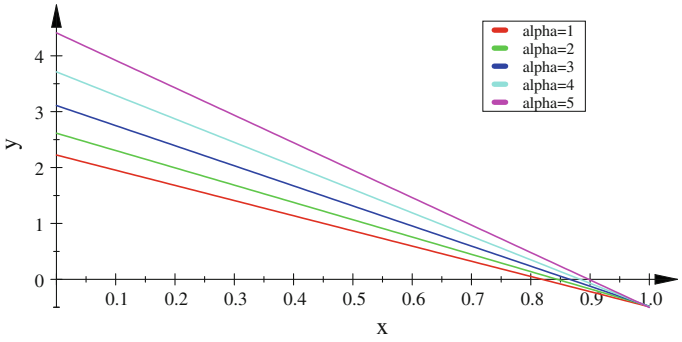

Fig. 4 Graph of function by $\mathbf{u}^{(n)} \times \mathbf{v}$ for alpha values 1, 2, ..., 5 and $n = 2$

$$p_i^{(eig)}(x, \alpha) \equiv \mathbf{u}^{(n)}(x, \alpha)^T \mathbf{x}_i(\alpha), \quad i = 1, \ldots, n \tag{1}$$

where

$$\mathbf{u}^{(n)}(x, \alpha) \equiv [u_1(x, \alpha) \ \ldots \ u_n(x, \alpha)]^T \tag{2}$$

The polynomials $p_i^{(eig)}(x, \alpha)$s are mutually orthonormal under the exponential weight on the interval [0, 1] as can be proven as long as the eigenvectors of the universal matrix $\mathbf{X}^{(n)}(\alpha)$ are normalized with respect to Frobenius norm (hence we assume that this normalization holds). However, these polynomials, although they lie in the weight function generating space, are not truely weight function since they change their signs throughout the interval [0, 1]. We call them "Eigenpolynomials of the Algebraic Independent Variable Operator Restriction on n–dimensional Supspace of the Hilbert Space" or briefly "nth Order Universal Eigenpolynomials".

Figure 4 depicts the universal eigenpolynomial corresponding to the smallest eigenvalue of the universal matrix, for different positive α values when $n = 2$, while Fig. 5

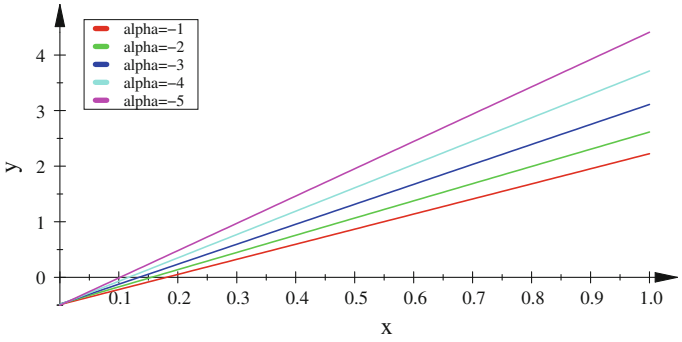

Fig. 5 Graph of function by $\mathbf{u}^{(n)} \times \mathbf{v}$ for alpha values $-1, -2, \ldots, -5$ and $n = 2$

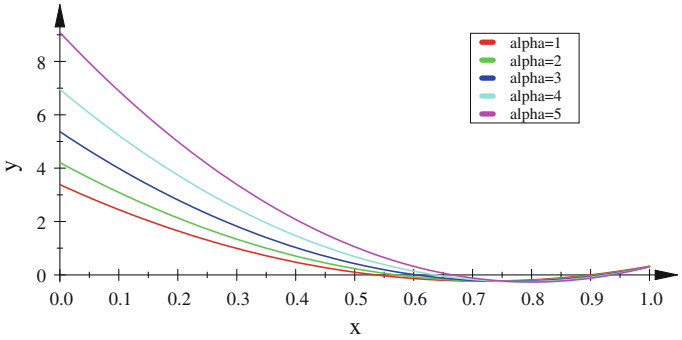

Fig. 6 Graph of function by $\mathbf{u}^{(n)} \times \mathbf{v}$ for alpha values $1, 2, \ldots, 5$ and $n = 3$

repeats the same thing by replacing the eigenpolynomial with the one corresponding to largest eigenvalue and negative α values. Since $n = 2$ the polynomials are of first degree and they intersect the abscissa axis between 0 and 1. This confirms that these are not weight functions although they lie in the weight function generating space. The intersection of these polynomials with the interval [0, 1] are accumulated at the right close to 1 for positive α values while negative α values cause an accumulation at the left close to 0.

Figures 6 and 7 shows the plots of the same polynomial by tracing the same philosophy for different (positive for the first, negative for the second) α values in the case where $n = 3$. In this case the degree of the polynomials is 2 and this gives not lines but curves. The zero accumulation nature is same as before. However, this time there are two zero accumulation points for each case such that positive α values produce accumulation points close to 1 while negative α values produce two accumulation points close to 0.

Figures 8 and 9 presents the similar situation for $n = 4$ where everything except the three accumulation points close to 1 or 0 for positive or negative α values, respectively. Another interesting feature is that the accumulation points are tightly related to the eigenvalues which are not corresponding to the used eigenvector in the construction of the considered eigenpolynomial.

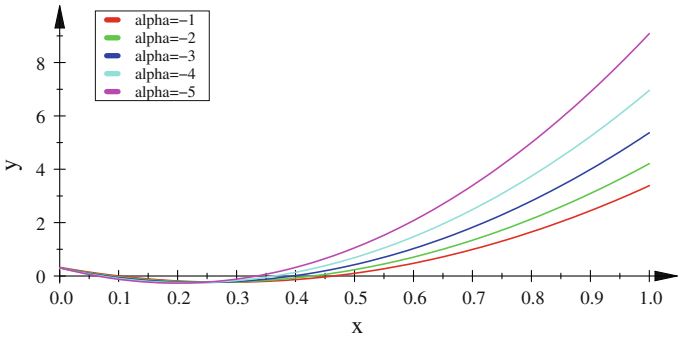

Fig. 7 Graph of function by $\mathbf{u}^{(n)} \times \mathbf{v}$ for alpha values $-1, -2, \ldots, -5$ and $n = 3$

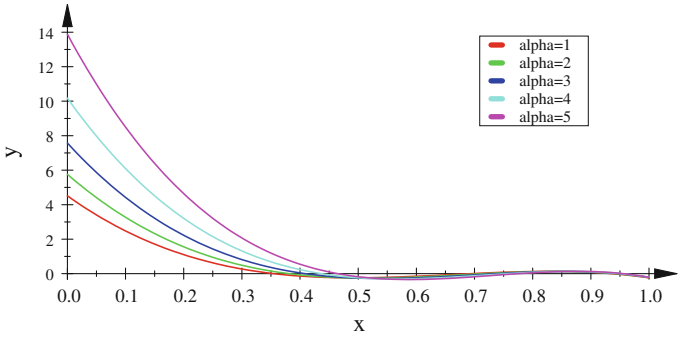

Fig. 8 Graph of function by $\mathbf{u}^{(n)} \times \mathbf{v}$ for alpha values $1, 2, \ldots, 5$ and $n = 4$

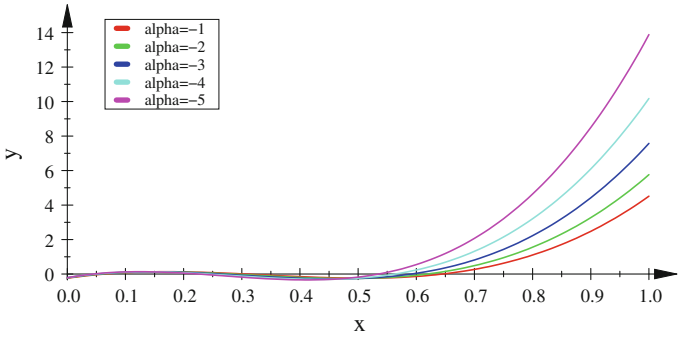

Fig. 9 Graph of function by $\mathbf{u}^{(n)} \times \mathbf{v}$ for alpha values $-1, -2, \ldots, -5$ and $n = 4$

3 Exponential weight function generators for one node case under more than one condition

In this paper, we are going to ultimately deal with the cases where the function values to be used in integration are given at more than one node. However, it is still better to go further in one node case to explain what we understand for the mechanism we want to construct. In the previous paper [10], we dealt with the case where just a single

condition was used to construct a weight function generating subspace. This may not be of course the only case. More than one conditions could be used to construct the space at the target to get more efficiency as more compatible conditions in fluctuation free integration or Gauss quadratures bring more numerical accuracy. The integral at the focus for this case will be taken in its simplest form as follows

$$\mathcal{I} \equiv \int_0^1 dx f(x) \tag{3}$$

where we have considered the unit constant weight function whose integral over [0, 1] is 1. In one node cases of the previous paper [10], we considered a function $s(x)$ in the Hilbert space of square integrable functions over the unit interval [0, 1] under unit constant weight function such that

$$\int_0^1 dx s(x)\,(x - x_{nd}) = 0 \tag{4}$$

where x_{nd} stands for the node where the value of the function to be integrated is given. We had investigated various structures for $s(x)$ and seen that what we had obtained was restricted to certain nodal intervals (for x_{nd}) which are narrower than the interval [0, 1] generally except for certain specific structures. One of these specific structures was related to exponential function $\exp(\alpha x)$ where α is a real parameter whose value uniquely determines the location of x_{nd} on the interval [0, 1] including the endpoints where α goes to minus or plus infinity for $x_{nd} = 0$ and $x_{nd} = 1$, respectively.

Now we are going to consider the case where the following conditions are imposed

$$\int_0^1 dx s(x)\,(x - x_{nd})^j = 0, \quad j = 1, \ldots, m \tag{5}$$

where the number of the conditions, m, can be called "Fluctuationlessness Level". To proceed as we did before we are going to assume the following structure for $s(x)$

$$s(x) \equiv \sum_{j=1}^{\infty} s_j w_e(x, \alpha)\, u_j(x), \quad w_e(x, \alpha) \equiv \frac{e^{\alpha x}}{e^{\alpha} - 1} \tag{6}$$

where the parameter α is arbitrary at this moment and will be determined accordingly to locate the node x_{nd} in the interval [0, 1] as desired. The other parameters, s_j coefficients, are also undetermined at this point and they reflect the dimension of the space under consideration. The symbols $u_j(x)$s stand for the polynomials which are orthonormalized from the power set $x^0, x^1, x^2, \ldots$over the unit interval [0, 1] and under the exponential weight function $w_e(x, \alpha)$ by conserving the ordering.

$s(x)$ above is in its most general form and can be used to find the functions which span the weight function generating space by imposing the m conditions given through the equations of (5). We obtain

$$\sum_{j=1}^{\infty}\left[\int_0^1 dx w_e(x,\alpha)\,u_j(x)\,(x-x_{nd})^k\right]s_j=0,\quad k=1,\ldots,m \tag{7}$$

which can be converted to the following set of equations by taking appropriate linear combinations amongst them

$$\sum_{j=1}^{\infty}\left[\int_0^1 dx w_e(x,\alpha)\,u_j(x)x^i\right]s_j=x_{nd}\sum_{j=1}^{\infty}\left[\int_0^1 dx w_e(x,\alpha)\,u_j(x)x^{i-1}\right]s_j,\quad i=1,\ldots,m \tag{8}$$

Now it is possible to replace x^{i-1} and x^i by $u_i(x)$ and $u_i(x)x$, respectively via the use of various appropriate linear combinations amongst the equations, to get

$$\sum_{j=1}^{\infty}\left[\int_0^1 dx w_e(x,\alpha)\,u_i(x)xu_j(x)\right]s_j=x_{nd}\sum_{j=1}^{\infty}\left[\int_0^1 dx w_e(x,\alpha)\,u_i(x)u_j(x)\right]s_j,\quad i=1,\ldots,m \tag{9}$$

which is equivalent to the following equations because of the $u_i(x)$ functions' orthonormalities

$$\sum_{j=1}^{\infty}\left[\int_0^1 dx w_e(x,\alpha)\,u_i(x)xu_j(x)\right]s_j=x_{nd}s_i,\quad i=1,\ldots,m \tag{10}$$

On the other hand, it is well known that the orthonormal polynomials satisfy three consecutive term recursive relations which imply

$$\sum_{j=1}^{m+1}\left[\int_0^1 dx w_e(x,\alpha)\,u_i(x)xu_j(x)\right]s_j=x_{nd}s_i,\quad i=1,\ldots,m \tag{11}$$

Therefore, all s_j coefficients for $j>m+1$ remain unaffected by the imposed conditions. This is quite natural and hence an expected result since each $u_j(x)$ which is a $(j-1)$th polynomial is orthogonal to all polynomials whose degrees are less than $m+1$ and therefore, to the monomials $(x-x_{nd})^k$ $(k=1,\ldots,m)$ for all js greater than $m+1$. The equations in (11) can be rewritten in the following matrix algebraic form

$$\left(\mathbf{X}^{(m)}(\alpha) - x_{nd}\mathbf{I}_m\right)\mathbf{s}^{(m)} = -\int_0^1 dx\, w_e(x,\alpha)\, u_m(x) x u_{m+1}(x) s_{m+1}\mathbf{e}_m \tag{12}$$

where $\mathbf{X}^{(m)}(\alpha)$ and $\mathbf{I}_m$ stand for the matrix representation of the independent variable over the subspace spanned by $u_1(x)$, …, $u_m(x)$ and the identity matrix of m dimension, respectively while the vector $\mathbf{s}^{(m)}$ is composed of the elements $s_1, \ldots, s_m$, respectively. $\mathbf{e}_m$ denotes the mth cartesian unit vector whose only nonzero element, 1, is located at the bottom most position.

Now the coefficient matrix at the left hand side of (12) is invertible if x_{nd} does not belong to the set of the $\mathbf{X}^{(m)}(\alpha)$'s eigenvalues, enabling us to express all s_is for i values from 1 to m inclusive. In that case the smallest degree polynomial in the weight function generating subspace has the degree of $m-1$. For its construction we can first write

$$\mathbf{s}^{(m)} = -\int_0^1 dx\, w_e(x,\alpha)\, u_m(x) x u_{m+1}(x) s_{m+1}\left[\mathbf{X}^{(m)}(\alpha) - x_{nd}\mathbf{I}_m\right]^{-1}\mathbf{e}_m \tag{13}$$

and then define

$$\mathbf{u}^{(m)}(x) \equiv [u_1(x)\ \cdots\ u_m(x)]^T. \tag{14}$$

These lead us to get

$$\begin{aligned} P_{pr}(x,\alpha) &\equiv \sum_{i=1}^{m+1} s_i u_i(x) = \mathbf{u}^{(m)}(x)^T \mathbf{s}^{(m)} + s_{m+1}u_{m+1}(x) \\ &= \left\{u_{m+1}(x) - \beta_m \mathbf{u}^{(m)}(x)^T\left[\mathbf{X}^{(m)}(\alpha) - x_{nd}\mathbf{I}_m\right]^{-1}\mathbf{e}_m\right\} s_{m+1} \end{aligned} \tag{15}$$

where

$$\beta_m \equiv \int_0^1 dx\, w_e(x,\alpha)\, u_m(x) x u_{m+1}(x) \tag{16}$$

We call $P_{pr}(x)$ "The Principal Nonsingular Basis Polynomial"of the weight generating subspace. We choose the value of s_{m+1} to get 1 value for the integral of this polynomial over the interval [0, 1] and under the weight $w_e(x,\alpha)$. This polynomial produces $(m+1)$ number of linearly independent mth degree polynomials for appropriately chosen $(m+1)$ different α values. This means that the unit constant function can be expressed as a linear combination of these polynomials. That is,

$$1 = \sum_{i=1}^{m+1} c_i P_{pr}(x,\alpha_i) \tag{17}$$

where the linear combination coefficients, c_is are determined to exactly satisfy the equation and the result is unique because of the linear independence (mentioned above) of the principal polynomials. This urges us to write

$$\int_0^1 dx f(x) = \sum_{i=1}^{m+1} c_i \int_0^1 dx P_{pr}(x, \alpha_i) f(x) = \sum_{i=1}^{m+1} c_i \int_0^1 dx S_{pr}(x, \alpha_i) \frac{f(x)}{w_e(x, \alpha)},$$
$$S_{pr}(x, \alpha_i) \equiv w_e(x, \alpha) P_{pr}(x, \alpha_i) \tag{18}$$

where we call $S_{pr}(x, \alpha_i)$ "Principal Basis Function"of the weight function genera ting subspace. If we use the one node fluctuation free integration for the function $f(x)/w_e(x, \alpha)$ then we can arrive at

$$\int_0^1 dx f(x) = \left(\sum_{i=1}^{m+1} \frac{c_i}{w_e(x_{nd})} \right) f(x_{nd}) \tag{19}$$

We call this formula "mth Order One Node Principal Fluctuation Free Integration". The sum multiplying $f(x_{nd})$ value can be considered as the weight constant of the fluctuation free integration. What we expect is the better numerical efficiency as the order, m, grows.

4 Numerical examples

Suppose that we will find the integral of $\exp(\beta x)$ in [0, 1] interval. Tables 1, 2, 3 give the comparison of exact integral and approximate integral of this function for different positive α and β values and for $x_{nd} = 0, 0.5, 1$, respectively for $m = 2$. Tables 4, 5, 6 give the comparison of exact integral and approximate integral of this function for different negative α and β values and for $x_{nd} = 0, 0.5, 1$, respectively for $m = 2$.

We can summarize the results as follows:

- To get good approximation α values must be chosen close to β value. Although we expect better numerical efficiency as m grows, it becomes difficult to choose appropriate αs.
- For positive β values, as β becomes larger, x_{nd} values closer to 1 gives better results.

Table 1 Comparison of real and approximate integral of $\exp(\beta x)$ for $x_{nd} = 0$ for positive α and β values

β	α_1	α_2	α_3	Exact integral	Approx. integral	Relative error
1	0.95	1	1.05	1.718281828	1.718344686	−0.00003658149625
4	3.95	4	4.05	13.39953751	13.40022851	−0.00005156929666
7	6.95	7	7.05	156.5190226	156.5301629	−0.00007117512637
10	9.95	10	10.05	2202.546579	2202.832245	−0.0001296976757

Table 2 Comparison of real and approximate integral of $\exp(\beta x)$ for $x_{nd} = 0.5$ for positive α and β values

β	α_1	α_2	α_3	Exact integral	Approx. integral	Relative error
1	0.95	1	1.05	1.718281828	1.718248721	0.00001926757134
4	3.95	4	4.05	13.39953751	13.39914832	0.00002904472514
7	6.95	7	7.05	156.5190226	156.5134321	0.00003571765411
10	9.95	10	10.05	2202.546579	2202.500961	0.00002071152089

Table 3 Comparison of real and approximate integral of $\exp(\beta x)$ for $x_{nd} = 1$ for positive α and β values

β	α_1	α_2	α_3	Exact integral	Approx. integral	Relative error
1	0.95	1	1.05	1.718281828	1.718354118	−0.00004207108447
4	3.95	4	4.05	13.39953751	13.39988794	−0.00002615242243
7	6.95	7	7.05	156.5190226	156.5224553	−0.00002193151009
10	9.95	10	10.05	2202.546579	2202.573521	−0.00001223178381

Table 4 Comparison of real and approximate integral of $\exp(\beta x)$ for $x_{nd} = 0$ for negative α and β values

β	α_1	α_2	α_3	Exact integral	Approx. integral	Relative error
−1	−0.95	−1	−1.05	0.6321205588	0.6321460318	−0.00004029770964
−4	−3.95	−4	−4.05	0.2454210903	0.2454284272	−0.00002989507438
−7	−6.95	−7	−7.05	0.142726874	0.1427299585	−0.00002161133314
−10	−9.95	−10	−10.05	0.09999546001	0.0999969181	−0.00001458163251

Table 5 Comparison of real and approximate integral of $\exp(\beta x)$ for $x_{nd} = 0.5$ for negative α and β values

β	α_1	α_2	α_3	Exact integral	Approx. integral	Relative error
−1	−0.95	−1	−1.05	0.6321205588	0.6321087504	0.00001868058994
−4	−3.95	−4	−4.05	0.2454210903	0.2454147695	0.00002575496297
−7	−6.95	−7	−7.05	0.142726874	0.1427223402	0.00003176559685
−10	−9.95	−10	−10.05	0.09999546001	0.09999056797	0.00004892255402

Table 6 Comparison of real and approximate integral of $\exp(\beta x)$ for $x_{nd} = 1$ for negative α and β values

β	α_1	α_2	α_3	Exact integral	Approx. integral	Relative error
−1	−0.95	−1	−1.05	0.6321205588	0.6321460014	−0.00004024955129
−4	−3.95	−4	−4.05	0.2454210903	0.2454334784	−0.00005047699139
−7	−6.95	−7	−7.05	0.142726874	0.1427365128	−0.00006753300603
−10	−9.95	−10	−10.05	0.09999546001	0.1000040009	−0.00008541276607

Table 7 Comparison of real and approximate integral of $\sin(\beta x)$ for $x_{nd} = 1$ for different α and β values

β	α_1	α_2	α_3	Exact integral	Approx. integral	Relative error
1	1.01	1.2	1.9	0.4596976941	0.4596321427	0.00014259671
2	0.2	0.45	1.0	0.7080734183	0.701230198	0.009664563104
3	−5.95	−1.005	0.1	0.6633308322	0.665693645	0.003562042

– For negative β values, as β becomes larger in absolute value, x_{nd} values closer to 0 gives better results.

For other functions like $\sin(\beta x)$, it is not so easy to find appropriate α values. Table 7 gives some results for $\sin(\beta x)$. Find ing a formal way to find appropriate α values is not in the scope of this paper.

For gauss function $\exp(((x - x_d)/\epsilon)^2)$, to get good approximation, x_{nd} must be chosen close to x_d. But as ϵ becomes smaller, which means that the function becomes steeper at x_d, it is not possible to get good approximation. But for large values of ϵ, it is possible to get good results. For example for $x_d = 0.5, \epsilon = 0.6$, if we choose $x_{nd} = 0.5$ and α values as $-0.03. - 3.5, 0.03$, we obtain approximate integral as 0.806335056 where exact integral is equal to 0.8097354431 in [0, 1] interval. This gives the relative error as 0.004199380249.

5 Characteristic subspace spanned by eigenpolynomials

The analysis in third section does not cover all possibilities for the weight function generating subspace. As we mentioned at the beginning of that section, all $u_j(x)$ polynomials whose degrees are greater than or equal to $(m+1)$ satisfy m annihilation conditions mentioned at the beginning of third section because of their orthonormalities and they are of course linearly independent of the principal polynomial. We call those polynomials, $u_j(x)$s $(j \geq m+1)$ "Secondary Regular Components". We could equivalently call them "Unaffecting Terms" or something like that.

There is one other possibility to solve (10) beyond the inversion of the coefficient matrix appearing there. There is almost nothing to prevent us for taking 0 value for s_{m+1} there. If we do so then we obtain

$$\left(\mathbf{X}^{(m)}(\alpha) - x_{nd}\mathbf{I}_m\right)\mathbf{s}^{(m)} = \mathbf{0} \tag{20}$$

to get the vector $\mathbf{s}^{(m)}$. However, this time we can obtain a nontrivial solution only when x_{nd} belongs to the spectrum of the matrix $\mathbf{X}^{(m)}(\alpha)$. If this happens to be then the vector $\mathbf{s}^{(m)}$ should be the universal matrix eigenvector corresponding to the eigenvalue which has been taken by x_{nd}. This is of course a great limitation but an undeniable existence of another basis function possibility for the weight function generating sub space. If we denote the eigenvalues and the eigenvectors of the universal matrix $\mathbf{X}^{(m)}(\alpha)$ by $\xi_i(\alpha)$ and $\mathbf{x}_i(\alpha)$, respectively (we have deliberately shown the α (exponential parameter) dependence here since we will explicitly use it later) then we can write

$$\mathbf{s}_i^{(m)} = \mathbf{x}_i(\alpha), \quad i = 1, \ldots, m \tag{21}$$

which urges us to define

$$P_i^{(eig)}(x, \alpha) \equiv \mathbf{s}_i^{(m)^T} \mathbf{u}^{(m)}(x) \quad i = 1, \ldots, m \tag{22}$$

which are $m - 1$ degree polynomials. Since these basis functions are indirectly corresponding to the eigenvectors of the universal matrix $\mathbf{X}^{(m)}(\alpha)$ we call them "Eigenpolynomials ". These do not span the weight function generating subspace or its some subspace since they have no weight function factor at this moment. However, we can define

$$S_i^{(eig)}(x, \alpha) \equiv P_i^{(eig)}(x, \alpha) \quad i = 1, \ldots, m \tag{23}$$

which are the desired "Eigenbasis Functions" for our exponential factor including weight function generating subspace.

Although there are m linearly independent eigenvector they can not be directly used in a linear combination to represent our unit constant weight function since each of them corresponds to a different eigenvalue and therefore to a different node. We can bypass this problem using a tricky way. If we consider a specific nodal value represented by x_{nd} as we did before, then we can seek a unique α exponential parameter value, say x_1, for which the greatest eigenvalue of the universal matrix $\mathbf{X}(\alpha_1)$ becomes x_{nd}. The next step is to seek an α value, say α_2, for which the second greatest eigenvalue of the universal matrix $\mathbf{X}(\alpha_2)$ becomes x_{nd}. Then this scheme is generalized con secutively. We try to find an α value, say α_i, in the ith step, for which the ith greatest eigenvalue of the universal matrix $\mathbf{X}(\alpha_i)$ becomes x_{nd}. Therefore, at the end, after m steps, we obtain the set of vectors, (we assume that the eigenvalues are in descending order), $\mathbf{x}(\alpha_1)_1, \mathbf{x}(\alpha_2)_2, \ldots, \mathbf{x}(\alpha_i)_i, \ldots, \mathbf{x}(\alpha_m)_m$ which are expected to be linearly independent. The scalar product of each vector in this set with the basis function vector $\mathbf{u}^{(m)}(x)$ defines an $(m-1)$th degree polynomial therefore, this polynomial set spans the m dimensional cartesian space in which the unit constant weight function lies. This enables us to write the following relations

$$p_i^{(eig)}(x, \alpha_i) \equiv \mathbf{u}^{(m)}(x, \alpha_i)^T \mathbf{x}(\alpha_i)_i, \quad i = 1, \ldots, m \tag{24}$$

$$1 = \sum_{i=1}^{m} c_i \, p_i^{(eig)}(x, \alpha_i) \tag{25}$$

which enables us to write

$$\int_0^1 dx f(x) = \sum_{i=1}^{m} c_i \int_0^1 dx p_i^{(eig)}(x, \alpha_i) \, f(x) \int_0^1 dx f(x)$$

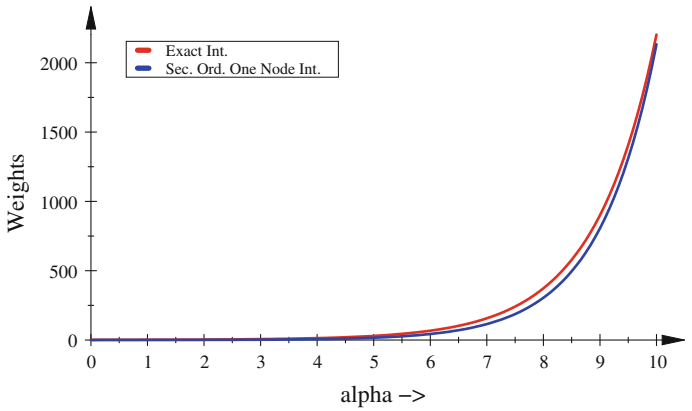

Fig. 10 Graph of second order one node fluctuation free integration based on eigenpolynomials

$$= \sum_{i=1}^{m} c_i \int_0^1 dx S_i^{(eig)}(x, \alpha_i) \frac{f(x)}{w_e(x, \alpha)} \tag{26}$$

The application of one node fluctuation free integration on the right hand side integrals gives

$$\int_0^1 dx f(x) = \left(\sum_{i=1}^{m} \frac{c_i}{w_e(x_{nd}, \alpha)} \right) f(x_{nd}) \tag{27}$$

This is the mth order one node fluctuation free integration based on eigenpolynomials. The Fig. 10 gives the comparison of exact and approximate integrals for second order one node fluctuation free integration based on eigenpolynomials. In this figure, α parameter is taken as 2 first and then α parameter is taken as 21.98. x_{nd} is equal to 0.84463397 which is one of the eigenvalues of the universal matrix $\mathbf{X}$ for each α values. Although the precision is very good in precision for $x_{nd} = 0.84463397$, this is somehow by chance since this node is almost the best nodal location for the one node integration of the exponential function. Despite we have not given here other x_{nd} values like 0.07, 0.6 which give worse approximations than 0.84463397. In other words, the choice of α parameters and also x_{nd} are function dependent. For monotonously increasing functions, x_{nd} values closer to 1 give better approximations. For monotonously decreasing functions, x_{nd} values closer to 0 produce better approximations.

6 Multinode fluctuation free integration

As we stated at the beginning, one of our ultimate goals was the derivation of multinode quadrature like formula. We do not attempt to give it here, since its construction based on one node integration is quite easy. If we assume that there are n given nodes

as interior points of [0, 1], then we can partition the total integration interval to subintervals such that their union gives [0, 1] and each subinterval contains exactly one node. Thus, total integal can be expressed as some of n subintegrals each of which has one integration node. Then each subinterval is converted to an integral on [0, 1] by an appropriate transformation. The next step is the approximate evaluation of each subintegral via one node fluctuation free integration under desiredly many imposed fluctuation removal conditions. What we expect is the increasing accuracy as n and number of imposed conditions grows unboundedly.

7 Concluding remarks

In this paper we developed a method which is "mth Order One Node Principal Fluctuation Free Integration". In this method we expect to get the better numerical efficiency as the order, m, grows. But, as m grows it gets very complicated to obtain appropriate α parameters, because these α parameters are dependent on the function to be integrated. Some poins we concluded can be given as follows: Suppose that we will find the integral of $\exp(\beta x)$ in [0, 1] interval. To get good approximation α values must be chosen close to β value. For positive β values, as β becomes larger, x_{nd} values closer to 1 give better results. For negative β values, as β becomes larger in absolute value, x_{nd} values closer to 0 give better results. For Gauss function $\exp(((x - x_d)/\epsilon)^2)$, to get good approximation, x_{nd} must be chosen close to x_d.

We also developed a method which is mth order one node fluctuation free integration based on eigenpolynomials. The Fig. 10 gives the comparison of real and approximate integrals for second order one node fluctuation free integration based on eigenpolynomials. Although it is universal when x_{nd} is prescribed the choice of α parameters and also x_{nd} become function dependent when they are left arbitrary and requested to be determined to produce higher precision. For monotonously increasing functions, x_{nd} values closer to 1 give better approximations, for monotonously decreasing functions, x_{nd} values closer to 0 give better approximations as we mentioned in the paper.

Here, we have given the fundamental issues of one node fluctuation free integration. Although the conceptual skeleton seems to be constructed efficiently, there remains many technical issues and problems to be solved to increase the efficiency and to minimize computational complexity for future works.

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